Vector Spaces and Introduction to Simultaneous Linear Equations

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Outline

- Review last lecture
- General vector spaces
- Inner products as generalization of dot products
- Norms as generalization of vector length
- Linear independence and basis sets
- Introduction to solutions of simultaneous equations

Review

- Vectors can be represented by components in terms of a basis set
- Dot product of two vectors: sum of product of components (orthogonal basis)
- Matrix basics: definition, equality, addition, multiplication, transpose, inverse
- Determinants evaluation of small sizes and general equation

Review Matrix Multiplication

- For matrix multiplication, **C** = **BA**
  - **B** has n rows and p columns
  - **A** has p rows and m columns
  - **C** has n rows and m columns

$$c_{ij} = \sum_{k=1}^{p} b_{ik} a_{kj} \quad (i = 1, n;\ j = 1, m)$$

- **B** is left matrix and **A** is right matrix
- In general **AB** ≠ **BA**
- Component c_{ij} in product matrix is product of row i components in left matrix with column j components in right matrix

Review Determinants

- Looks like a matrix but isn't a matrix
- Use | | instead of [ ] for borders
- A square array of numbers with a rule for computing a single value for the array

$$Det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$Det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{matrix}$$

Review General Determinants

- Evaluate determinant by any equation

$$Det\,\mathbf{A}_{(n\,x\,n)} = \sum_{i=1}^{n}(-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n}(-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

- M_{ij}, is size (n-1) determinant found by removing row i and column j from **A**
  - M_{ij} is minor determinant; C_{ij} is cofactor
- Can pick ***any*** row ***or any*** column
- Choose row or column with several zeros
- Numerical determinants found more efficiently by Gaussian elimination

Review Inverse of a Matrix

- For square matrix, **A**, the inverse, $\mathbf{A}^{-1}$, if it exists, gives $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- Find the components of $\mathbf{B} = \mathbf{A}^{-1}$, b_{ij}, from determinant of **A** and its cofactors

$$If\ \mathbf{B} = \mathbf{A}^{-1}, \qquad b_{ij} = \frac{C_{ji}}{Det(\mathbf{A})} = (-1)^{i+j}\frac{M_{ji}}{Det(\mathbf{A})}$$

- Use this formula to get algebraic equations for components of inverse matrix not for numerical analysis

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Vector Spaces

- In mechanics a vector is a physical quantity with magnitude and direction expressed as two or three components
- General vectors have n components and may not represent a physical quantity
- Vector spaces have simple rules which focus on representing all vectors in the space by a set of basis vectors
- This is a unifying concept for many of the topics covered in ME 501AB

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Rules for Vector Spaces

- Vector spaces have simple rules to allow many examples of vectors
- If **x** and **y** are vectors in the space then **x** + **y** is also a vector in the space.
- The addition operation is commutative and associative. That is, **x** + **y** = **y** + **x** and **x** + **y** + **z** = (**x** + **y**) + **z** = **x** + (**y** + **z**).
- The space contains a null element, **0**, such that **x** + **0** = **0** + **x** = **x**.

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More Simple Rules

- For each vector, **x**, in the space there is another vector, **–x**, such that **x** + (**–x**) = **0**
- Vectors can be multiplied by scalars.
- Scalars (and vector components) may be real or complex
- Complex numbers use i where $i^2 = -1$
- For complex scalars, $|\alpha|^2 = \alpha^*\alpha$, where $\alpha^* = u - iv = re^{-i\theta}$ is the complex conjugate of $\alpha = u + iv = re^{i\theta}$

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Multiplication by Scalars

x and **y** are vectors in the space and α and β are (real or complex) scalars

$\alpha\mathbf{x}$, $\beta\mathbf{x}$, $\alpha\mathbf{y}$, and $\beta\mathbf{y}$ are all vectors in the space

$(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

$(\alpha + \beta)\mathbf{y} = \alpha\mathbf{y} + \beta\mathbf{y}$

$\alpha\beta\mathbf{x} = (\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$

$\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

$1\mathbf{x} = \mathbf{x}$

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Vector Norms

- Norm of a vector, **x**, expressed as $||\mathbf{x}||$,
- Measure of the size of the vector.
- Generalization of the usual definition of vector length
- Any definition must satisfy
  - $||\alpha\mathbf{x}|| = |\alpha|\ ||\mathbf{x}||$ (If complex, $|\alpha|^2 = \alpha^*\alpha$)
  - $||\mathbf{x}|| > 0$ if $\mathbf{x} \neq \mathbf{0}$
  - $||\mathbf{x}|| = 0$ if $\mathbf{x} = \mathbf{0}$
  - $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$

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The q Norm

Definition: $\|\mathbf{x}\|_q = \left[\sum |x_i|^q\right]^{1/q}$

- Generalizes notion of vector length
- Vector length is the "two norm", $\|\mathbf{x}\|_2$
- Other common norms
  - one norm: sum of absolute values
  - infinity norm: the element with maximum absolute value.

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From Dot to Inner Products

- Inner product generalizes vector dot product
- Inner product notation (**x**,**y**)
- Inner products satisfy the following
  - $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})^*$
  - $(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$
  - $(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$
  - $(\mathbf{x}, \mathbf{x}) > 0$ unless $\mathbf{x} = \mathbf{0}$
  - $(\mathbf{x}, \beta\mathbf{y}) = \beta^*(\mathbf{x}, \mathbf{y})$

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Inner Product Definitions

- For two conventional real vectors, $[x_1\ x_2\ x_3\ \ldots\ x_n]$ and $[y_1\ y_2\ y_3\ \ldots\ y_n]$, the inner product is $\Sigma x_i y_i$
- For two column vectors, **x** and **y**, we can express the inner product as $\mathbf{x}^T\mathbf{y}$
- For two row vectors, **x** and **y**, we can express the inner product as $\mathbf{x}\mathbf{y}^T$
- We can also define inner products as integrals of two functions

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Linear Combinations

- We can form linear combinations of any number, k, vectors in the space. The linear combination is defined in terms of a set of k scalars, $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that our linear combination is given by the equation

$$\alpha_1\mathbf{x}_{(1)} + \alpha_2\mathbf{x}_{(2)} + \cdots + \alpha_k\mathbf{x}_{(k)} = \sum_{i=1}^{k} \alpha_i\mathbf{x}_{(i)}$$

Notation: $\mathbf{x}_{(i)}$ is one vector in a set of vectors, not a vector component like x_i

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Linear Dependence

- A set of vectors ***linearly dependent*** if the following equation holds, where at least one of the α_i is not equal to zero.

$$\alpha_1\mathbf{x}_{(1)} + \alpha_2\mathbf{x}_{(2)} + \cdots + \alpha_k\mathbf{x}_{(k)} = \sum_{i=1}^{k} \alpha_i\mathbf{x}_{(i)} = \mathbf{0}$$

- If one of the $\alpha_i \neq 0$ we can divide by that α (say α_1) and get a dependent equation

$$\mathbf{x}_{(1)} = -\left(\frac{\alpha_2}{\alpha_1}\right)\mathbf{x}_{(2)} - \left(\frac{\alpha_3}{\alpha_1}\right)\mathbf{x}_{(3)} - \cdots - \left(\frac{\alpha_k}{\alpha_1}\right)\mathbf{x}_{(k)}$$

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Linear Independence

- A set of vectors that is not **linearly dependent** is said to be **linearly independent**
- In a linearly ***independent*** set of vectors, we cannot do this
- Next slides consider examples of sets of vectors that linearly independent and linearly dependent

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Linear (In)dependence in 3D

- Consider usual unit vectors
  - **i** = [1 0 0], **j** = [0 1 0], and **k** = [0 0 1]
  - α_1**i** + α_2**j** + α_3**k** = [α_1 α_2 α_3]
  - [α_1 α_2 α_3] = **0** = [0 0 0] only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$
  - Thus, as expected, these three vectors are linearly independent
- Replace **i** by **m** = [1 1 1], giving the set
  - **m** = [1 1 1], **j** = [0 1 0], and **k** = [0 0 1]
  - α_1**m** + α_2**j** + α_3**k** = [α_1 $\alpha_1+\alpha_2$ $\alpha_1+\alpha_3$]
  - How can [α_1 $\alpha_1+\alpha_2$ $\alpha_1+\alpha_3$] = **0** = [0 0 0]?

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Linear (In)dependence in 3D II

- α_1**m** + α_2**j** + α_3**k** = **0** = [0 0 0], only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$
  - Thus **m**, **j**, and **k** are linearly independent
- **Another set: m** = [1 1 1], **j** = [0 1 0], and **p** = [1 0 1]
  - α_1**m** + α_2**j** + α_3**p** = α_1[1 1 1] + α_2[0 1 0] + α_3[1 0 1] = [$\alpha_1+\alpha_3$ $\alpha_1+\alpha_2$ $\alpha_1+\alpha_3$]
  - [$\alpha_1+\alpha_3$ $\alpha_1+\alpha_2$ $\alpha_1+\alpha_3$] = [0 0 0] if $\alpha_1 = c$ $\alpha_2 = \alpha_3 = -c$
  - **m**, **j**, and **p** are not linearly independent because **m** - **j** - **p** = **0** ($\alpha_1 = 1$ $\alpha_2 = \alpha_3 = -1$)

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n-dimensional vector space

- Has a set of n linearly independent vectors
- Has no sets of n+1 (or more) linearly independent vectors
- Any vector in an n-dimensional space can be represented by a linearly independent combination of n vectors.
- A set of n linearly independent vectors is called a **basis set** and is said to **span the space**

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Orthogonal Vectors

- Two vectors whose inner product equals zero are ***orthogonal***.
- *I. e.*, **x** and **y** are orthogonal if (**x**, **y**) = 0.
- n vectors, $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, … , $\mathbf{e}_{(n)}$, are mutually orthogonal if the inner product of any unlike pair of vectors vanishes
- *I. e.*, if ($\mathbf{e}_{(i)}$, $\mathbf{e}_{(j)}$) = 0 for any i and j ($i \neq j$), the set of vectors is orthogonal
- Orthogonal vectors linearly independent

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Orthonormal Vectors

- For ***orthonormal*** vectors, $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, … , $\mathbf{e}_{(n)}$, the inner product of any unlike pair of vectors is zero and the inner product of like vectors is one.
- Orthonormal set: ($\mathbf{e}_{(i)}$, $\mathbf{e}_{(j)}$) = δ_{ij}
- In mechanics we use $\mathbf{e}_{(1)}$,= **i**, $\mathbf{e}_{(2)}$,= **j**, and $\mathbf{e}_{(3)}$,= **k** as an orthonormal set; we know that ($\mathbf{e}_{(i)}$, $\mathbf{e}_{(j)}$) = δ_{ij} for this set

Kronecker delta $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

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Orthogonal to Orthonormal

- For an orthogonal set ($\mathbf{b}_{(i)}$, $\mathbf{b}_{(j)}$) = $a_i\delta_{ij}$ where a_i = ($\mathbf{b}_{(i)}$, $\mathbf{b}_{(i)}$)
- For an orthonormal set, ($\mathbf{e}_{(i)}$, $\mathbf{e}_{(j)}$) = δ_{ij}
- To convert an orthogonal set, $\mathbf{b}_{(i)}$ to an orthonormal set divide $\mathbf{b}_{(i)}$ by $(\mathbf{b}_{(i)}, \mathbf{b}_{(i)})^{1/2}$

$$\mathbf{e}_{(i)} = \frac{\mathbf{b}_{(i)}}{\sqrt{(\mathbf{b}_{(i)}, \mathbf{b}_{(i)})}}$$

$$(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = \left(\frac{\mathbf{b}_{(i)}}{\sqrt{(\mathbf{b}_{(i)}, \mathbf{b}_{(i)})}}, \frac{\mathbf{b}_{(j)}}{\sqrt{(\mathbf{b}_{(j)}, \mathbf{b}_{(j)})}} \right) = \frac{(\mathbf{b}_{(i)}, \mathbf{b}_{(j)})}{\sqrt{(\mathbf{b}_{(i)}, \mathbf{b}_{(i)})(\mathbf{b}_{(j)}, \mathbf{b}_{(j)})}} = \delta_{ij}$$

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Functions in Vector Spaces

- Functions such as $\sin(n\pi x/L)$ form a vector space in the region $0 \le x \le L$.
- The inner product, defined below, shows that this is a set of orthogonal functions

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm}$$

- The set of functions at the right is orthonormal $\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$
- Weight function sometimes used

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Simultaneous Equations

- The second column is an equivalent set of equations that is a linear combination of the equations in the first column

$3x_1 + 5x_2 = 13$ $6x_1 + 11x_2 = 28$	$3x_1 + 5x_2 = 13$ $x_2 = 2$	$x_1 = 1$ $x_2 = 2$
$3x_1 + 5x_2 = 13$ $6x_1 + 10x_2 = 26$	$3x_1 + 5x_2 = 13$ $0 = 0$	$x_1 = a \quad (any\ a)$ $x_2 = \frac{13-3a}{5}$
$3x_1 + 5x_2 = 13$ $6x_1 + 10x_2 = 25$	$3x_1 + 5x_2 = 13$ $0 = -1$	*No solution*

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Two Basic Ideas

- A set of simultaneous linear algebraic equations may have
  - A single (unique) solution
  - No solution
  - An infinite number of solutions
- A linear combination of any two equations can replace one of the equations and not change the solution

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Getting to a Matrix Form

- Example of 3 equations (3 unknowns)
  3x + 7y – 3z = 8
  2x – 4y + z = -3
  8x + 6y – 2z = 14
- How can we develop a general notation for N equations in N unknowns?
  - Call variables x_1, x_2, x_3 etc.
  - Call right hand side b_1, b_2, b_3, etc.
  - Call top row coefficients a_{11}, a_{12}, a_{13}, etc.
    - Coefficient of x_m in equation k is a_{km}

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Standard Form

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 +...+ a_{1N-1}x_{N-1} + a_{1N}x_N = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 +...+ a_{2N-1}x_{N-1} + a_{2N}x_N = b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 +...+ a_{3N-1}x_{N-1} + a_{3N}x_N = b_3$
… … … … … … … … … … … … … … … … … … …
$a_{N-1,1}x_1 + a_{N-1,2}x_2 + .. … … … …+ a_{N-1,N}x_N = b_{N-1}$
$a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 +… … … … ..+ a_{NN}x_N = b_N$

- Usual subscripts on a are $a_{row,column}$
- Row is equation and column is unknown, x_k
- N can be any number

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Compact Standard Forms

- **Ax = b** $\quad \sum_{j=1}^{N} a_{ij}x_j = b_i \quad i = 1,\ldots,N$
- Equations defined by data: N, a_{ij}, and b_i
- a_{ij} coefficients are a matrix, **A**
  - Use negative numbers for a_{ij} in place of subtraction
- Right-hand side, **b**, and unknowns, **x**, are column vectors (x_{j1}, b_{i1})
- Summation is just usual matrix multiplication formula

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Example in Standard Form

- Previous example of 3 equations (N = 3)
  - 3x + 7y – 3z = 8
  - 2x – 4y + z = -3
  - 8x + 6y – 2z = 14
- In standard form:
  - x is x_1, y is x_2, and z is x_3
  - $a_{11} = 3$, $a_{12} = 7$, $a_{13} = -3$, $b_1 = 8$
  - $a_{21} = 2$, $a_{22} = -4$, $a_{23} = 1$, $b_2 = -3$
  - $a_{31} = 8$, $a_{32} = 6$, $a_{33} = -2$, $b_3 = 14$

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Example in Standard Form

- Previous example of N = 3 equations

  3x + 7y – 3z = 8
  2x – 4y + z = -3
  8x + 6y – 2z = 14

- As **Ax** = **b**

$$\begin{bmatrix} 3 & 7 & -3 \\ 2 & -4 & 1 \\ 8 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 14 \end{bmatrix}$$

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Solving **Ax** = **b**

- Know **A** (all the a_{ij}) and **b** (all b_i)
- Want **x** (all the unknowns x_i)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

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General System for **Ax** = **b**

(n x m) (m x 1) (n x 1)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

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n equations and m unknowns?

- How can this be? We expect m = n
- First we have to see if the n equations are really independent equations
- Systems for m > n have an infinite number of solutions
- Systems for n > m can be solved in a least squares sense
  - Provide solution that has least error in inner product (**Ax** – **b**, **Ax** – **b**)

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Gauss Elimination

- Practical tool for obtaining solutions
- Analytical tool for determining linear dependence or independence
- Basic idea is to manipulate the equations (or data) to make them easier to solve without changing the results
- Systematically create zeros in lower left part of the equations (or data)

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Upper Triangular Form

- Convert original set of equations to

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \cdots & \alpha_{1n-1} & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \cdots & \alpha_{2n-1} & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & \cdots & \cdots & \alpha_{3n-1} & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{n-1n-1} & \alpha_{n-1n} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

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Gauss Elimination III

- Upper triangular form on previous slide is easily solved by back substitution
- $x_n = \beta_n/\alpha_{nn}$
- $x_{n-1} = (\beta_{n-1} - \alpha_{n-1\,n}x_n)/\alpha_{n-1\,n-1}$, *et cetera*
- General equation for back substitution

$$x_i = \frac{\beta_i - \sum_{j=i+1}^{n} \alpha_{ij} x_j}{\alpha_{ii}} \qquad i = n-1, n-2, \ldots, 1$$

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Quick Example

N = 3 equations

3x + 7y – 3z = 8
2x – 4y + z = -3
8x + 6y – 2z = 14

Matrix form:
Ax = b

$$\begin{bmatrix} 3 & 7 & -3 \\ 2 & -4 & 1 \\ 8 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 14 \end{bmatrix}$$

Subtract (2/3) Row 1 from Row 2
Subtract (8/3) Row 1 from Row 3

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Quick Example II

$$\begin{bmatrix} 3 & 7 & -3 \\ 2-\frac{2}{3}3 & -4-\frac{2}{3}7 & 1-\frac{2}{3}(-3) \\ 8-\frac{8}{3}3 & 6-\frac{8}{3}7 & -2-\frac{8}{3}(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -3-\frac{2}{3}8 \\ 14-\frac{8}{3}8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 & -3 \\ 0 & -\frac{26}{3} & 3 \\ 0 & -\frac{38}{3} & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -\frac{25}{3} \\ -\frac{22}{3} \end{bmatrix}$$

Subtract (38/26) times Row 2 from Row 3

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Quick Example III

$$\begin{bmatrix} 3 & 7 & -3 \\ 0 & -\frac{26}{3} & 3 \\ 0 & -\frac{38}{3} - \frac{-\frac{38}{3}}{-\frac{26}{3}}\left(-\frac{26}{3}\right) & 6 - \frac{-\frac{38}{3}}{-\frac{26}{3}}3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -\frac{25}{3} \\ -\frac{22}{3} - \frac{-\frac{38}{3}}{-\frac{26}{3}}\left(-\frac{25}{3}\right) \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 & -3 \\ 0 & -\frac{26}{3} & 3 \\ 0 & 0 & \frac{21}{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -\frac{25}{3} \\ \frac{63}{13} \end{bmatrix}$$

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Quick Example IV

$$\begin{bmatrix} 3 & 7 & -3 \\ 0 & -\frac{26}{3} & 3 \\ 0 & 0 & \frac{21}{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -\frac{25}{3} \\ \frac{63}{13} \end{bmatrix} \qquad x_3 = \frac{63/13}{21/13} = 3$$

$$x_2 = \frac{-25/3 - 3(3)}{-26/3} = \frac{-52/3}{-26/3} = 2$$

$$x_1 = \frac{8 - (-3)(3) - (7)(2)}{3} = \frac{3}{3} = 1$$

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